GENERAL PROBLEM OF SYNCHRONIZATION IN AN ALMOST CONSERVATIVE SYSTEM

(OBSHCHAIA ZADACHA O SINKHRONIZATSII V POCHTI KONSERVATIVNOI SISTEME)

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R.F. NAGAEV
(Leningrad)

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The attempt is made to examine the problem of synchronization in an almost conservative system of weakly coupled dynamic objects with a maximum of common positions. The need for such generalizations is due to the desire to work out a single method for analyzing numerous manifestations of the synchronization phenomenon in nature and technology [1]. For this purpose a sufficiently general classification of types of conservative couplings is introduced. This classification, apparently, can be extended to quite a broad class of systems of interacting objects. The physical meaning and method of introduction of the small parameter of coupling is discussed in detail. This permits a clear presentation of specifics of the system. It is shown that the character of the stable synchronous regime in the system depends in a strong manner on the type of coupling. Conditions are clarified under which the generalized integral criterion of stability of the synchronous regime in the system is valid [2]. In conclusion a degenerate special case is examined which in particular leads to a quasi-linear formulation of the problem.

1. Types of couplings and basic dynamic characteristics of the system. A system of k dynamic objects with weak mutual linkages will be examined. The motion of an arbitrary ith object in the system will be characterized by the $l \times 1$ column vector of characteristic partial generalized coordinates

$$q_i = (q_{i1}, \ldots, q_{l_i})$$

The method of introduction of characteristic coordinates of the object is assumed to be independent of the character of couplings and in this sense not completely arbitrary. For this reason characteristic coordinates completely preserve their physical significance even in the total absense of couplings between the objects. The form of dependence of dynamic characteristics of the object, which are determined only by its characteristic generalized coordinates and velocities, on all these quantities is invariant with respect to the type of couplings. Thus, the "characteristic" kinetic energy

$$T_i = {}^1/_2\, \mathbf{q}_i^{\cdot\prime} \mathbf{A}_i (\mathbf{q}_i)\, \mathbf{q}_i^{\cdot} \qquad (1.1)$$

where $\mathbf{A}_i$ is symmetric ($\mathbf{A}_i = \mathbf{A}_i'$) "inertial" $l_i \times l_i$ matrix, and the "characteristic" potential energy of the object

$$\Pi_i = \Pi_i(\mathbf{q}_i) \tag{1.2}$$

have the form of corresponding characteristics of the object in the absence of couplings. Here and in the following, transposition of the corresponding matrix is designated by a prime.

At the appearance of interaction, objects in the system acquire in the general case some additional mobility so that for description of their motions in the intercoupled system it is necessary to specify as well $m \times 1$ column vector of additional generalized coordinates

$$x = (x_1, \ldots, x_m)$$

In this case the general kinetic energy and generalized potential energy of objects in the system are written in the form

$$T^* = \sum_{i=1}^{k} T_i + \Delta T^*, \qquad \Pi^* = \sum_{i=1}^{k} \Pi_i + \Delta\Pi^* \tag{1.3}$$

Here

$$\Delta T^* = \sum_{i=1}^{k} \mathbf{q}_i^{\cdot\prime} \mathbf{A}_{im}(\mathbf{x}, \mathbf{q})\,\mathbf{x}^{\cdot} + \tfrac{1}{2}\mathbf{x}^{\cdot\prime}\mathbf{A}_m(\mathbf{x}, \mathbf{q})\,\mathbf{x}^{\cdot}, \qquad \Delta\Pi^* = \mathbf{x}'\mathbf{C}(\mathbf{q}, \nu t) + \ldots \tag{1.4}$$

are additional kinetic and potential energies of objects (*). It will be assumed that some part of external $2\pi/\nu$ periodic excitation which in general can be transmitted to the objects by means of coupling elements, is such that it can be included into $\Delta\Pi^*$.

Further, the concept of a supporting body or a supporting system of bodies having m degrees of freedom will be associated with the total of additional generalized coordinates $x_1, \ldots, x_m$. It will be assumed here that coordinates $x_1, \ldots, x_m$ are absolute in the sense that they completely describe the motion of the supporting system. Interaction which is produced with the aid of the supporting system, is always associated with the appearance of new degrees of freedom in the intercoupled system of objects. These will be called couplings of the first kind.

The kinetic and potential energy of the supporting system have the form

$$T^{(1)} = {}^1\!/_2 \mathbf{x}^{\cdot\prime}\mathbf{M}_m(\mathbf{x})\,\mathbf{x}^{\cdot}, \qquad \Pi^{(1)} = {}^1\!/_2 \mathbf{x}'\mathbf{C}_m\mathbf{x} + \ldots \tag{1.5}$$

The supporting system may have distributed parameters and as a result of this it may have an infinite denumerable number of degrees of freedom. However, in such a case the possibility of introducing normal coordinates x_1, $x_2, \ldots$ is always provided for the description of motion of the supporting system. In other words, for any m, in particular for $m = \infty$, coordinates $x_1, \ldots, x_m$ can be selected such that symmetrical $m \times m$ matrices $\mathbf{M}_{m\,|\,x=0}$ and $\mathbf{C}_m$ will take the form

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*) In the expression for $\Delta\Pi^*$ and Expression (1.5) for $\Pi^{(1)}$ only the first terms of expansions of corresponding quantities in power series with respect to $x_1, \ldots, x_m$ are written out.

$$\mathbf{M}_m\big|_{x=0} = \operatorname{diag}(m^{(1)}, \ldots, m^{(m)}), \qquad \mathbf{C}_m = \operatorname{diag}(c^{(1)}, \ldots, c^{(m)}) \quad (1.6)$$

The process of solution of the problem in the presence of a supporting system with distributed parameters is described in [2] for a fairly general case, however, in general, it lacks a rigorous mathematical foundation.

Appearance of couplings of a different nature, or of couplings of the second kind between objects is not necessarily connected with an increase of degrees of freedom in the intercoupled system. However, in the general case, for a description of dynamics of their elements it is necessary in addition to $\mathbf{x}$ and $\mathbf{q}_1, \ldots, \mathbf{q}_k$ to specify as well $n \times 1$ column vector $\mathbf{y}$ with generalized coordinates $y_1, \ldots, y_n$, which naturally are not needed for the description of motion of the supporting system and objects.

Kinetic energy and generalized potential energy of couplings of the second kind will be correspondingly

$$T^{(2)} = \tfrac{1}{2}\mathbf{y}^{\cdot\prime}\mathbf{N}_n(\mathbf{y}, \mathbf{q}, \mathbf{x})\,\mathbf{y}^{\cdot} + \sum_{i=1}^{n}\mathbf{q}_i^{\cdot\prime}\mathbf{N}_{in}(\mathbf{y}, \mathbf{q}, \mathbf{x})\,\mathbf{y}^{\cdot} + \sum_{i,\,j=1}^{k}\tfrac{1}{2}\mathbf{q}_i^{\cdot\prime}\mathbf{N}_{ij}(\mathbf{y}, \mathbf{q}, \mathbf{x})\,\mathbf{q}_j^{\cdot} +$$

$$+\,\mathbf{x}^{\cdot\prime}\mathbf{N}_{mn}(\mathbf{y}, \mathbf{q}, \mathbf{x})\,\mathbf{y}^{\cdot} + \sum_{i=1}^{k}\mathbf{q}_i^{\cdot\prime}\mathbf{N}_{im}(\mathbf{y}, \mathbf{q}, \mathbf{x})\,\mathbf{x}^{\cdot} + \tfrac{1}{2}\mathbf{x}^{\cdot\prime}\mathbf{N}_m(\mathbf{y}, \mathbf{q}, \mathbf{x})\,\mathbf{x}^{\cdot} \quad (1.7)$$

$$\Pi^{(2)} = \Pi^{(2)}(\mathbf{y}, \mathbf{q}, \mathbf{x}, \nu t)$$

Thus, an external single-frequency excitation can be transmitted to objects only by means of couplings of the first and second kind.

2. Criterion for weakness of interaction between objects. As weakness of interaction or, which is the same thing, as weakness of couplings we will understand the possibility in the analysis of the system to introduce effectively a small positive parameter of coupling μ such that for $\mu = 0$ the objects in the system may be regarded as isolated. Here from the very beginning we will agree on considering the values of dynamic and kinematic chatacteristics of motion of objects in the region of interest to us as quantities of the order of 1, i.e.

$$\mathbf{q} = O(1), \qquad \mathbf{A} = O(1), \qquad \mathbf{C} = O(1)$$

which in a certain sense determines equivalent position of objects in the system.

Actions transmitted by means of couplings are small. Consequently, after superposition of couplings, the general dynamic characteristics of the system change insignificantly. General kinetic and potential energy of the system satisfy the relationships (2.1)

$$T = T^* + T^{(1)} + T^{(2)} = \sum_{i=1}^{k} T_i + O(\mu), \quad \Pi = \Pi^* + \Pi^{(1)} + \Pi^{(2)} = \sum_{i=1}^{k}\Pi_i + O(\mu)$$

Taking into account the essential positiveness of terms in relationships (2.1), we shall note the conditions under which they can be satisfied.

1) Since components $l_i \times m$ of matrix A_{im} characterize inertial properties of the ith object as a whole, and consequently are of the order of 1, the estimate

$$\Delta T^* = O(\mu)$$

will turn out to be valid only if the change of coordinates of the supporting system is small in the process of motion, i.e.

$$\mathbf{x} = \mu \boldsymbol{\xi} \qquad (\xi = O(1)) \tag{2.2}$$

2) For dynamic characteristics of the supporting system to be small it is sufficient that

$$\mathbf{M}_m = \frac{\mathbf{M}_m^\circ}{\mu} + O(1), \qquad \mathbf{C}_m = \frac{\mathbf{C}_m^\circ}{\mu} + O(1) \tag{2.3}$$

where components $m \times m$ of quadratic matrices $\mathbf{M}_m^\circ$ and $\mathbf{C}_m^\circ$ are constant.

3) Inertial and force characteristics of elements of couplings of the second kind are small so that

$$\mathbf{N}(\mathbf{y}, \mathbf{q}, \mathbf{x}) = \mu \mathbf{N}^\circ(\mathbf{y}, \mathbf{q}) + O(\mu^2)$$

$$T^{(2)} = \mu \left(\tfrac{1}{2} \mathbf{y}^{\cdot\prime} \mathbf{N}_n^\circ \mathbf{y}^{\cdot} + \sum_{i=1}^{k} \mathbf{q}_i^{\cdot\prime} \mathbf{N}_{im}^\circ \mathbf{y}^{\cdot} + \tfrac{1}{2} \sum_{i,j=1}^{k} \mathbf{q}_i^{\cdot\prime} \mathbf{N}_{ij}^\circ \mathbf{q}_j^{\cdot} \right) + O(\mu^2) \tag{2.4}$$

$$\Pi^{(2)} = \mu \pi^{(2)}(\mathbf{y}, \mathbf{q}, \nu t) + O(\mu^2)$$

Thus, while the weakness of couplings of the first kind is predetermined by the smallness of oscillations of the supporting body, the weakness of couplings of the second kind is due to the smallness of generalized impulses which correspond to them. As a consequence of what was mentioned above, the fact of linearity of equations of motion of the system with respect to oscillatory coordinates of the supporting body is established (with accuracy to quantities of the order of μ^2).

Finally, the assumption of conservative nature of interaction between objects in the system is introduced (with the same degree of accuracy). Generalized nonpotential forces $\mathbf{Q}_i = (Q_{i1}, \ldots, Q_{il_i})$ in coordinates of objects, then have partial character and are small. The appropriateness of the last assumption was clarified in [3].

3. Equations of motion of the system in the form of Routh and the generating system of the problem of synchronization. Column vectors of generalized impulses of objects are introduced into the examination

$$\mathbf{p}_i = \frac{\partial T}{\partial \mathbf{q}_i^{\cdot}} = \mathbf{A}_i \mathbf{q}_i^{\cdot} + \mu \ldots \tag{3.1}$$

Transforming systems (3.1), we will have

$$\mathbf{q}_i^{\cdot} = \mathbf{A}_i^{-1} \mathbf{p}_i + \mu \mathbf{q}_i^{\cdot(1)}(\mathbf{q}, \mathbf{p}, \boldsymbol{\xi}, \boldsymbol{\xi}^{\cdot}, \mathbf{y}, \mathbf{y}^{\cdot}) + \mu^2 \ldots \tag{3.2}$$

The following analysis, in which the elucidation of the physical meaning of some quantities entering into the equation of motion of the system will have great importance, is most conveniently carried out by putting together certain equations of Routh which are ordinarily used in the presence of cyclic coordinates. For this purpose we introduce the function of Routh

$$R = T - \sum_{i=1}^{k} \mathbf{p}_i{}'\mathbf{q}_i{}^{\cdot} = -\tfrac{1}{2}\sum_{i=1}^{k} \mathbf{p}_i{}'\mathbf{A}_i{}^{-1}\mathbf{p}_i + \mu\Big(\tfrac{1}{2}\boldsymbol{\xi}^{\cdot\prime}\mathbf{M}_m{}^{\circ}\boldsymbol{\xi}^{\cdot} + \sum_{i=1}^{k} \mathbf{p}_i{}'\mathbf{A}_i{}^{-1}\mathbf{A}_{im}^{\circ}\boldsymbol{\xi}^{\cdot} +$$

$$+ \tfrac{1}{2}\mathbf{y}^{\cdot\prime}\mathbf{N}_n{}^{\circ}\mathbf{y}^{\cdot} + \sum_{i=1}^{k} \mathbf{p}_i{}'\mathbf{A}_i{}^{-1}\mathbf{N}_{in}^{\circ}\mathbf{y}^{\cdot} + \tfrac{1}{2}\sum_{i,\,j=1}^{k} \mathbf{p}_i{}'\mathbf{A}_i{}^{-1}\mathbf{N}_{ij}^{\circ}\mathbf{A}_j{}^{-1}\mathbf{p}_j\Big) + \mu^2 \ldots \quad (3.3)$$

and the kinetic potential of Routh

$$L_R = R - \Pi = -\sum_{i=1}^{k} H_i + \mu L_0 + \mu^2 \ldots, \qquad L_0 = \Delta L^* + L^{(1)} + L^{(2)} \quad (3.4)$$

In relationships (3.4) with accuracy to quantities of the order of μ^2

$$H_i = \tfrac{1}{2}\mathbf{p}_i{}'\mathbf{A}_i{}^{-1}\mathbf{p}_i + \Pi_i \quad (3.5)$$

is the "characteristic" energy (Hamiltonian function) of the ith partial object,

$$\mu\Delta L^* = \mu\Big(\sum_{i=1}^{k} \mathbf{p}_i{}'\mathbf{A}_i{}^{-1}\mathbf{A}_{im}{}^{\circ}\boldsymbol{\xi}^{\cdot} - \mathbf{C}'\boldsymbol{\xi}\Big) \quad (3.6)$$

is the additional kinetic potential of objects due to small oscillations of the supporting system,

$$\mu L^{(1)} = \mu\left({}^1/_2\,\boldsymbol{\xi}^{\cdot\prime}\mathbf{M}_m{}^{\circ}\boldsymbol{\xi}^{\cdot} - {}^1/_2\boldsymbol{\xi}'\mathbf{C}_m{}^{\circ}\boldsymbol{\xi}\right) \quad (3.7)$$

is the kinetic potential of the supporting system,

$$\mu L^{(2)} = \mu\Big(\tfrac{1}{2}\mathbf{y}^{\cdot\prime}\mathbf{N}_n{}^{\circ}\mathbf{y}^{\cdot} + \sum_{i=1}^{k} \mathbf{p}_i{}'\mathbf{A}_i{}^{-1}\mathbf{N}_{in}{}^{\circ}\mathbf{y}^{\cdot} + \tfrac{1}{2}\sum_{i,\,j=1}^{k} \mathbf{p}_i{}'\mathbf{A}_i{}^{-1}\mathbf{N}_{ij}{}^{\circ}\mathbf{A}_j{}^{-1}\mathbf{p}_j - \pi^{(2)}\Big) \quad (3.8)$$

is the kinetic potential of elements of couplings of the second kind.

Equations of motion for a coupled system of objects in the form of Routh can be obtained for example by means of the usual variational method on the basis of central equation of Lagrange [4]. They have the following form:

$$\mathbf{q}_i{}^{\cdot} - \frac{\partial H_i}{\partial \mathbf{p}_i} = -\mu\frac{\partial}{\partial \mathbf{p}_i}(\Delta L^* + L^{(2)}) + \mu^2 \ldots$$

$$\mathbf{p}_i{}^{\cdot} + \frac{\partial H_i}{\partial \mathbf{q}_i} = \mu\Big[\mathbf{Q}_i(\mathbf{q}_i, \mathbf{p}_i) + \frac{\partial}{\partial \mathbf{q}_i}(\Delta L^* + L^{(2)})\Big] + \mu^2 \ldots \qquad (i = 1, \ldots, k) \quad (3.9)$$

$$\mathbf{M}_m{}^{\circ}\boldsymbol{\xi}^{\cdot\cdot} + \mathbf{C}_m{}^{\circ}\boldsymbol{\xi} + \Big(\frac{d}{dt}\frac{\partial}{\partial \boldsymbol{\xi}^{\cdot}} - \frac{\partial}{\partial \boldsymbol{\xi}}\Big)\Delta L^* + \mu \ldots = 0$$

$$\mu\Big(\frac{d}{dt}\frac{\partial}{\partial \mathbf{y}^{\cdot}} - \frac{\partial}{\partial \mathbf{y}}\Big)L^{(2)} + \mu^2 \ldots = 0$$

The generating system of the problem (*) in coordinates of objects disintegrates naturally into k self-contained coservative subsystems

$$\mathbf{q}_i{}^{\circ\cdot} = \frac{\partial H_i}{\partial \mathbf{p}_i{}^{\circ}}, \qquad \mathbf{p}_i{}^{\circ\cdot} = -\frac{\partial H_i}{\partial \mathbf{q}_i{}^{\circ}} \qquad (i = 1, \ldots, k) \quad (3.10)$$

*) In the selection of the generating system, just as in [3], it is necessary to keep in mind that for some terms of the order of μ may be extracted from the left parts of equations of objects (3.9) and can be related to $L^{(2)}$.

each of which in some region G_i of partial phase space $(\mathbf{q}_i, \mathbf{p}_i)$ permits only one single-valued analytical first integral of energy and has an orbitally stable particular solution

$$\mathbf{q}_i^\circ = \mathbf{q}_i^\circ(\varphi_i, c_i), \qquad \mathbf{p}_i^\circ = \mathbf{p}_i^\circ(\varphi_i, c_i) \tag{3.11}$$

This solution is 2π-periodic in the generalized sense [5] with respect to the characteristic rapidly rotating phase

$$\varphi_i = \omega_i t + \alpha_i \tag{3.12}$$

Solution (3.11) depends in a continuous manner on an arbitrary phase displacement α_i and parameter c_i which on trajectories of the solution under examination is connected in a continuous one-to-one corresponding manner with the constant of energy

$$H_i(q_i^\circ, p_i^\circ) = h_i(c_i) \tag{3.13}$$

The frequency (angular velocity) of the solution can vary along the trajectory of solution (3.11) within G_i and within the limits of the range

$$\omega_i \in [\omega_i^{(1)}, \omega_i^{(2)}] \tag{3.14}$$

The dependence of the frequency of solution (3.11) on the constant of energy, which is given parametrically as

$$h_i = h_i(c_i), \qquad \omega_i = \omega_i(c_i)$$

will be called the skeletal curve of the ith object as applied to the solution under examination.

The possibility of a system entering into synchronization in case of sufficiently weak couplings (μ is sufficiently small) is predetermined by the presence of a synchronous generating approximation [3], i.e. by the presence of Equations

$$\omega_1 = \ldots = \omega_k = \nu \tag{3.15}$$

In other words, the intersection of frequency ranges of generating objects (band of transmission of the system) must not be empty and must include in it the frequency of external excitation

$$[\omega^{(1)}, \omega^{(2)}] = \bigcap_{i=1}^{k} [\omega_i^{(1)}, \omega_i^{(2)}] \in \nu \tag{3.16}$$

Equations of motion of elements of couplings in the generating approximation permit a stable $2\pi/\nu$ periodic solution

$$\mathbf{u} = \mathbf{u}(\tau, c_1, \ldots, c_k, \alpha_1, \ldots, \alpha_k) \qquad (\tau = \nu t, \ \mathbf{u} = (\xi_1^\circ, \ldots, \xi_m^\circ, y_1^\circ, \ldots, y_n^\circ)) \tag{3.17}$$

Quantities $c_1, \ldots, c_k$ in Expression (3.17) are determined uniquely from relationships (3.15). The presence of a limited solution (3.17) implies in particular that the frequencies of free small oscillations of the supporting systems, determinable from Equation

$$|\mathbf{C}_m^\circ - \lambda_s^2 \mathbf{M}_m^\circ| = 0 \tag{3.18}$$

satisfy the following condition:

$$\lambda_s \neq r\nu \qquad (s = 1, \ldots, m, \ r = 1, 2, \ldots)$$

4. General case of essentially nonlinear objects. By means of ordinary substitution only of variables of coupling ξ, $\xi^{\cdot}$, y and $y^{\cdot}$ the system of equations (3.9) can be reduced to system

$$2\left(\sum_{i=1}^{N} l_i + m + n\right)$$

of the first order, solved with respect to derivatives.

In the same way the equation of motion of objects essentially does not change with accuracy to quantities of the order of μ inclusive. Equations of motion in coordinates of coupling do not contain any singularities because of initial assumptions. Known theorems on existence and stability of periodic solutions for sufficiently small value of parameter [2 and 5] turn out to be applicable directly to system (3.9).

The most general case is examined below where all objects in the generating approximation are substantially nonisochronous in the sense that everywhere within G_i the dependence $\omega_i = \omega_i(c_i)$ is essential and

$$\omega_i^{(2)} - \omega_i^{(1)} = O(1), \qquad \text{or} \qquad d\omega_i / dc_i = O(1) \tag{4.1}$$

the frequency range of the object is not small.

In this case parameters $c_1, \ldots, c_k$ which characterize the synchronous generating approximation are determined uniquely from relationships (3.15). The lack of isolation is associated exclusively with the presence of arbitrary phase displacements $\alpha_1, \ldots, \alpha_k$. The condition for the existence of a synchronous regime in the system in this case coincides with the condition for the presence of real solutions of the following system of equations:

$$P_i = F_i + R_i(\alpha_1, \ldots, \alpha_k) = 0 \qquad (i = 1, \ldots, k) \qquad \left(F_i = \frac{1}{2\pi}\int_0^{2\pi} Q_i' q_i^{\cdot} d\tau\right) \tag{4.2}$$

which is the power average over a period of nonpotential forces of the ith object in the generating approximation. The functions of phase displacements R are reduced to the following form after some simple transformations including integration by parts:

$$R_i = \frac{1}{2\pi}\int_0^{2\pi}\left(\mathbf{p}_i^{\cdot\cdot}\frac{\partial}{\partial \mathbf{p}_i} + \mathbf{q}_i^{\cdot\cdot}\frac{\partial}{\partial \mathbf{q}_i}\right)\left(\Delta L^* + L^{(2)}\right)d\tau = \frac{\partial \Lambda}{\partial \alpha_i} \quad (i = 1, \ldots, k) \tag{4.3}$$

The quantity

$$\Lambda(\alpha_1, \ldots, \alpha_k) = \frac{\nu}{2\pi}\int_0^{2\pi} L_0\, d\tau \tag{4.4}$$

will be called the action integral of the coupling in the generating approximation.

It is noted that over the period the average values of inherent dynamic characteristics of partial objects do not depend on phase displacements in the generating approximation; therefore

$$\frac{2\pi}{\nu} R_i = \frac{1}{\mu} \frac{\partial}{\partial \alpha_i} \int_0^{2\pi} L_R \, d\tau = \frac{1}{\mu} \frac{\partial}{\partial \alpha_i} \int_0^{2\pi} L \, d\tau \tag{4.5}$$

where L is the general kinetic potential of the system.

We shall perform a transformation of the virial over the equation of small fluctuations of the supporting system in the generating approximation. For this purpose it is multiplied by the line-vector $\xi^{\circ\prime}$ from the left; then it is not difficult to arrive at the following scalar equality

$$\frac{d}{dt}(\xi^{\circ\prime} \mathbf{M}_m{}^\circ \xi^{\circ\prime}) - 2L^{(1)} + \frac{d}{dt} \sum_{i=1}^{k} \mathbf{p}_i{}^{\circ\prime} \mathbf{A}_i{}^{-1} \mathbf{A}_{im} \xi^\circ - \Delta L^* = 0$$

Averaging the last relationship over the period we will have

$$\frac{\nu}{2\pi} \int_0^{2\pi} (2L^{(1)} + \Delta L^*) \, d\tau = 0 \tag{4.6}$$

Thus, for any value of phase displacements $\alpha_1, \ldots, \alpha_k$, over the period the average additional kinetic potential of objects is equal to twice the action integral of the supporting system taken with the opposite sign. Keeping in mind (4.6) we shall write the expression for the action integral of coupling (4.4) in the form

$$\Lambda = \frac{\nu}{2\pi} \int_0^{2\pi} (L^{(2)} - L^{(1)}) \, d\tau \tag{4.7}$$

Stability of the synchronous regime under investigation can be studied in the first approximation for example in such a way as it was done in [2]. Then it turns out that for small scale stability of synchronous motions of objects it is necessary (but not sufficient) that roots of Equation

$$\left| \frac{1}{k_i} \frac{\partial^2 \Lambda}{\partial \alpha_i \partial \alpha_j} - \delta_{ij} \varkappa \right| = 0 \tag{4.8}$$

be real and negative. In Equation (4.8) the quantity

$$k_i = \left. \frac{(dh_i/dc_i)}{(d\omega_i/dc_i)} \right|_{\omega_i = \nu} \tag{4.9}$$

which characterizes the speed of energy change of the ith partial object with change in the frequency of its motion in the zone of stabilization of synchronous frequency of the system, will be called the coefficient of steepness of the skeletal curve of the object. Sufficient conditions for stability of the synchronous regime of the system, which for the case of essentially different objects have, apparently, quite nontrivial character, were found in [3] for the particular case of objects with one degree of freedom in the system with couplings of the second kind.

Just as in preceding investigations [1 to 3], the condition for existence and stability of a synchronous regime in a system of identical or, more exactly, almost identical purely conservative objects ($k_1 = \ldots = k_k = k$) is reduced to a requirement of extremum of the action integral of coupling Λ with respect to generating phase displacements. The character of the extremum (maximum or minimum) is determined in the first place by the kind of coupling and in the second place by the sign of the coefficient of steepness of objects.

Thus, the character of phasing in stable synchronous motions of objects depends essentially on the nature of coupling (see (4.7)). Under certain conditions couplings of different kinds can completely or partially cancel the action of each other. As a result of this, the system will fall out of synchronism due to the presence of nonpotential nonuniformities of random character.

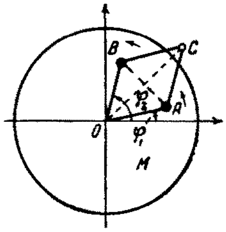
Fig. 1.

Let us examine a most simple example which illustrates the properties described above.

Let a system be given consisting of two identical almost conservative mechanical vibrators rotating in one direction around one and the same axis. Coupling of the first kind between vibrators is accomplished by means of massive supporting body of mass M. Coupling of the second kind is accomplished by means of mass m_0 placed at the apex C of the hinged pivoted rhombus $OACB$.

Couplings between vibrators are actually small if the following relationships are sufficiently smalll

$$\mu_1 = m / M, \qquad \mu_2 = m_0 / m$$

where m are equal masses of vibrators. Here the synchronous motion of vibrators in the generating approximation bears the character of uniform rotation with angular velocity ν

$$\varphi_1^\circ = \nu t + \alpha_1, \qquad \varphi_2^\circ = \nu t + \alpha_2$$

Kinetic energies of elements of coupling of the first and second kind, respectively, are independent of time with accuracy to quantities of the order of μ^2 and are equal to

$$T^{(1)} = \mu_1 2m\varepsilon^2\nu^2 \left[1 + \cos(\alpha_1 - \alpha_2)\right], \qquad T^{(2)} = \mu_2 m\varepsilon^2\nu^2 \left[1 + \cos(\alpha_1 - \alpha_2)\right]$$

The action integral of coupling in the generating approximation (4.7) is written in the form

$$\mu\Lambda = m\varepsilon^2\nu^3 (\mu_2 - 2\mu_1) \left[1 + \cos(\alpha_1 - \alpha_2)\right]$$

Simple analysis leads to the following conclusions.

1. In the absence of couplings of the second kind there exists a stable, out-of-phase regime of synchronous rotation of vibrators. For this regime the kinetic energy of the supporting body is minimal.

2. In the absence of couplings of the first kind it is opposite. The in-phase regime is stable and the kinetic energy of mass m_0 is maximal.

3. In the presence of couplings of both kinds the system falls out of synchronization if the approximate relationship $2\mu_1 \approx \mu_2$ holds.

5. Isochronism in the generating approximation. The special particular case is examined briefly below, when the geberating system (3.10) can be selected such that the dependence of frequency on energy disappears, the frequency ranges of objects reduce to a point and the conditions of existence of synchronous generating approximation (3.15) are satisfied identically. This case, which in particular leads to quasi-linear formulation of the problem, is most simple for analysis and probably because of this has been well studied previously for a series of actual examples.

Nonisolation of synchronous generating approximation is associated here not only with uncertainty of phase displacements $\alpha_1, \ldots, \alpha_k$, but also with arbitrariness in the selection of values of energy parameters $c_1, \ldots, c_k$.

Equations for investigation of parameters of synchronous generating solution, the number of which correspondingly increases by a factor of two, are written in the form

$$F_i + \frac{\partial\Lambda}{\partial\alpha_i} = 0, \qquad \Phi_i + \frac{\partial\Lambda}{\partial c_i} = 0 \qquad (i = 1, \ldots, k) \tag{5.1}$$

where $F_i(c_i)$ and $\Lambda(\alpha_1, \ldots, \alpha_k; c_1, \ldots, c_k)$ have their previous physical meaning and are computed from Equations (4.4) and (4.9), while

$$\Phi_i(c_i) = \frac{1}{2\pi}\int_0^{2\pi} Q_i' \frac{\partial q_i^0}{\partial c_i}\, d\tau \qquad (5.2)$$

For asymptotic stability of the synchronous regime under examination it is necessary and sufficient that all roots of Equation

$$\begin{vmatrix} \dfrac{\partial^2\Lambda}{\partial\alpha_i\partial\alpha_j} - \delta_{ij}\varkappa & \dfrac{\partial^2\Lambda}{\partial\alpha_i\partial c_j} + \dfrac{dF_i}{dc_i}\delta_{ij} \\ \dfrac{\partial^2\Lambda}{\partial\alpha_i\partial c_i} & \dfrac{\partial^2\Lambda}{\partial c_i\partial c_j} + \left(\dfrac{d\Phi_i}{dc_i} - \varkappa\right)\delta_{ij} \end{vmatrix} = 0 \qquad (5.3)$$

satisfy the condition $\operatorname{Re}\varkappa < 0$.

In the case of a purely conservative system (under the assumption that nonpotential forces in the system have higher degree of smallness) we arrive again at the formulation of the integral criterion of stability. In other words, for existence and asymptotic stability of synchronous regime in such a system the presence of a strict minimum of the potential function Λ with respect to variables $\alpha_1, \ldots, \alpha_k$ and $c_1, \ldots, c_k$ is necessary and sufficient. In this case due to symmetry of determinant (5.3) all its roots turn out to be real.

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